CHAPTER 8

DIFFERENTIATION OF REAL VALUED FUNCTIONS

Differential calculus is a highly geometric subject—a fact which is not always made entirely clear in elementary texts, where the study of derivatives as numbers often usurps the place of the fundamental notion of linear approximation. The contemporary French mathematician Jean Dieudonné comments on the problem in chapter 8 of his magisterial multivolume treatise on the *Foundations of Modern Analysis*[3]

> ... the fundamental idea of calculus [is] the "local" approximation of functions by *linear* functions. In the classical teaching of Calculus, this idea is immediately obscured by the accidental fact that, on a one-dimensional vector space, there is a one-to-one correspondence between linear forms and numbers, and therefore the derivative at a point is defined as a *number* instead of a *linear form*. This slavish subservience to the shibboleth of numerical interpretation at any cost becomes much worse when dealing with functions of several variables ...

The goal of this chapter is to display as vividly as possible the geometric underpinnings of the differential calculus. The emphasis is on "tangency" and "linear approximation", not on number.

8.1. THE FAMILIES $\mathfrak{O}$ AND $\mathfrak{o}$

8.1.1. Notation. Let $a \in \mathbb{R}$. We denote by $\mathcal{F}_a$ the family of all real valued functions defined on a neighborhood of a. That is, f belongs to $\mathcal{F}_a$ if there exists an open set U such that $a \in U \subseteq \operatorname{dom} f$.

Notice that for each $a \in \mathbb{R}$, the set $\mathcal{F}_a$ is closed under addition and multiplication. (We define the sum of two functions f and g in $\mathcal{F}_a$ to be the function $f + g$ whose value at x is $f(x) + g(x)$ whenever x belongs to $\operatorname{dom} f \cap \operatorname{dom} g$. A similar convention holds for multiplication.)

Among the functions defined on a neighborhood of zero are two subfamilies of crucial importance; they are $\mathfrak{O}$ (the family of "*big-oh*" functions) and $\mathfrak{o}$ (the family of "*little-oh*" functions).

8.1.2. Definition. A function f in $\mathcal{F}_0$ belongs to $\mathfrak{O}$ if there exist numbers $c > 0$ and $\delta > 0$ such that

$$|f(x)| \leq c\,|x|$$

whenever $|x| < \delta$.

A function f in $\mathcal{F}_0$ belongs to $\mathfrak{o}$ if for every $c > 0$ there exists $\delta > 0$ such that

$$|f(x)| \leq c\,|x|$$

whenever $|x| < \delta$. Notice that f belongs to $\mathfrak{o}$ if and only if $f(0) = 0$ and

$$\lim_{h \to 0} \frac{|f(h)|}{|h|} = 0\,.$$

8.1.3. Example. Let $f(x) = \sqrt{|x|}$. Then f belongs to neither $\mathfrak{O}$ nor $\mathfrak{o}$. (A function belongs to $\mathfrak{O}$ only if in some neighborhood of the origin its graph lies between two lines of the form $y = cx$ and $y = -cx$.)

8.1.4. Example. Let $g(x) = |x|$. Then g belongs to $\mathfrak{O}$ but not to $\mathfrak{o}$.

8.1.5. Example. Let $h(x) = x^2$. Then h is a member of both $\mathfrak{O}$ and $\mathfrak{o}$.