Much of the elementary theory of differential calculus rests on a few simple properties of the families $\mathfrak{O}$ and $\mathfrak{o}$. These are given in propositions 8.1.8–8.1.14.

8.1.6. Definition. A function $L\colon \mathbb{R} \to \mathbb{R}$ is LINEAR if

$$L(x + y) = L(x) + L(y)$$

and

$$L(cx) = cL(x)$$

for all x, y, $c \in \mathbb{R}$. The family of all linear functions from $\mathbb{R}$ into $\mathbb{R}$ will be denoted by $\mathfrak{L}$.

The collection of linear functions from $\mathbb{R}$ into $\mathbb{R}$ is not very impressive, as the next problem shows. When we get to spaces of higher dimension the situation will become more interesting.

8.1.7. Example. A function $f\colon \mathbb{R} \to \mathbb{R}$ is linear if and only if its graph is a (nonvertical) line through the origin.

PROOF. Problem.

CAUTION. Since linear functions must pass through the origin, straight lines are not in general graphs of linear functions.

8.1.8. Proposition. *Every member of $\mathfrak{o}$ belongs to $\mathfrak{O}$; so does every member of $\mathfrak{L}$. Every member of $\mathfrak{O}$ is continuous at 0.*

PROOF. Obvious from the definitions. □

8.1.9. Proposition. *Other than the constant function zero, no linear function belongs to $\mathfrak{o}$.*

PROOF. Exercise. (Solution Q.8.1.)

8.1.10. Proposition. *The family $\mathfrak{O}$ is closed under addition and multiplication by constants.*

PROOF. Exercise. (Solution Q.8.2.)

8.1.11. Proposition. *The family $\mathfrak{o}$ is closed under addition and multiplication by constants.*

PROOF. Problem.

The next two propositions say that the composite of a function in $\mathfrak{O}$ with one in $\mathfrak{o}$ (in either order) ends up in $\mathfrak{o}$.

8.1.12. Proposition. *If $g \in \mathfrak{O}$ and $f \in \mathfrak{o}$, then $f \circ g \in \mathfrak{o}$.*

PROOF. Problem.

8.1.13. Proposition. *If $g \in \mathfrak{o}$ and $f \in \mathfrak{O}$, then $f \circ g \in \mathfrak{o}$.*

PROOF. Exercise. (Solution Q.8.3.)

8.1.14. Proposition. *If $\phi, f \in \mathfrak{O}$, then $\phi f \in \mathfrak{o}$.*

PROOF. Exercise. (Solution Q.8.4.)

Remark. The preceding facts can be summarized rather concisely. (Notation: $\mathcal{C}_0$ is the set of all functions in $\mathcal{F}_0$ which are continuous at 0.)

(1) $\mathfrak{L} \cup \mathfrak{o} \subseteq \mathfrak{O} \subseteq \mathcal{C}_0\,.$

(2) $\mathfrak{L} \cap \mathfrak{o} = 0\,.$

(3) $\mathfrak{O} + \mathfrak{O} \subseteq \mathfrak{O}\,; \qquad \alpha\,\mathfrak{O} \subseteq \mathfrak{O}\,.$

(4) $\mathfrak{o} + \mathfrak{o} \subseteq \mathfrak{o}\,; \qquad \alpha\,\mathfrak{o} \subseteq \mathfrak{o}\,.$

(5) $\mathfrak{o} \circ \mathfrak{O} \subseteq \mathfrak{o}\,.$

(6) $\mathfrak{O} \circ \mathfrak{o} \subseteq \mathfrak{o}\,.$

(7) $\mathfrak{O} \cdot \mathfrak{O} \subseteq \mathfrak{o}\,.$