8.1.15. Problem. Show that $\mathfrak{O} \circ \mathfrak{O} \subseteq \mathfrak{O}$. That is, if $g \in \mathfrak{O}$ and $f \in \mathfrak{O}$, then $f \circ g \in \mathfrak{O}$. (As usual, the domain of $f \circ g$ is taken to be $\{x \colon g(x) \in \operatorname{dom} f\}$.)

8.2. TANGENCY

The fundamental idea of differential calculus is the local approximation of a "smooth" function by a translate of a linear one. Certainly the expression "local approximation" could be taken to mean many different things. One sense of this expression which has stood the test of usefulness over time is "tangency". Two functions are said to be tangent at zero if their difference lies in the family $\mathfrak{o}$. We can of course define tangency of functions at an arbitrary point (see project 8.2.12 below); but for our purposes, "tangency at 0" will suffice. All the facts we need to know concerning this relation turn out to be trivial consequences of the results we have just proved.

8.2.1. Definition. Two functions f and g in $\mathcal{F}_0$ are TANGENT AT ZERO, in which case we write $f \simeq g$, if $f - g \in \mathfrak{o}$.

8.2.2. Example. Let $f(x) = x$ and $g(x) = \sin x$. Then $f \simeq g$ since $f(0) = g(0) = 0$ and $\lim\limits_{x \to 0} \dfrac{x - \sin x}{x} = \lim\limits_{x \to 0} \left(1 - \dfrac{\sin x}{x}\right) = 0$.

8.2.3. Example. If $f(x) = x^2 - 4x - 1$ and $g(x) = \left(3x^2 + 4x - 1\right)^{-1}$, then $f \simeq g$.

PROOF. Exercise. (Solution Q.8.5.)

8.2.4. Proposition. *The relation "tangency at zero" is an equivalence relation on $\mathcal{F}_0$.*

PROOF. Exercise. (Solution Q.8.6.)

The next result shows that at most one linear function can be tangent at zero to a given function.

8.2.5. Proposition. *Let $S, T \in \mathfrak{L}$ and $f \in \mathcal{F}_0$. If $S \simeq f$ and $T \simeq f$, then $S = T$.*

PROOF. Exercise. (Solution Q.8.7.)

8.2.6. Proposition. *If $f \simeq g$ and $j \simeq k$, then $f + j \simeq g + k$, and furthermore, $\alpha f \simeq \alpha g$ for all $\alpha \in \mathbb{R}$.*

PROOF. Problem.

Suppose that f and g are tangent at zero. Under what circumstances are $h \circ f$ and $h \circ g$ tangent at zero? And when are $f \circ j$ and $g \circ j$ tangent at zero? We prove next that sufficient conditions are: h is linear and j belongs to $\mathfrak{O}$.

8.2.7. Proposition. *Let $f, g \in \mathcal{F}_0$ and $T \in \mathfrak{L}$. If $f \simeq g$, then $T \circ f \simeq T \circ g$.*

PROOF. Problem.

8.2.8. Proposition. *Let $h \in \mathfrak{O}$ and $f, g \in \mathcal{F}_0$. If $f \simeq g$, then $f \circ h \simeq g \circ h$.*

PROOF. Problem.

8.2.9. Example. Let $f(x) = 3x^2 - 2x + 3$ and $g(x) = \sqrt{-20x + 25} - 2$ for $x \leq 1$. Then $f \simeq g$.

PROOF. Problem.

8.2.10. Problem. Let $f(x) = x^3 - 6x^2 + 7x$. Find a linear function $T \colon \mathbb{R} \to \mathbb{R}$ which is tangent to f at 0.

8.2.11. Problem. Let $f(x) = |x|$. Show that there is no linear function $T \colon \mathbb{R} \to \mathbb{R}$ which is tangent to f at 0.