8.2.12. Problem. Let $T_a\colon x \mapsto x + a$. The mapping T_a is called TRANSLATION BY a. Note that it is *not* linear (unless, of course, $a = 0$). We say that functions f and g in $\mathcal{F}_a$ are TANGENT AT a if the functions $f \circ T_a$ and $g \circ T_a$ are tangent at zero.

(a) Let $f(x) = 3x^2 + 10x + 13$ and $g(x) = \sqrt{-20x - 15}$. Show that f and g are tangent at -2.
(b) Develop a theory for the relationship "tangency at a" which generalizes our work on "tangency at 0".

8.2.13. Problem. Each of the following is an abbreviated version of a proposition. Formulate precisely and prove.

(a) $\mathcal{C}_0 + \mathfrak{O} \subseteq \mathcal{C}_0$.
(b) $\mathcal{C}_0 + \mathfrak{o} \subseteq \mathcal{C}_0$.
(c) $\mathfrak{O} + \mathfrak{o} \subseteq \mathfrak{O}$.

8.2.14. Problem. Suppose that $f \simeq g$. Then the following hold.

(a) If g is continuous at 0, so is f.
(b) If g belongs to $\mathfrak{O}$, so does f.
(c) If g belongs to $\mathfrak{o}$, so does f.

8.3. LINEAR APPROXIMATION

One often hears that differentiation of a smooth function f at a point a in its domain is the process of finding the best "linear approximation" to f at a. This informal assertion is not quite correct. For example, as we know from beginning calculus, the tangent line at $x = 1$ to the curve $y = 4 + x^2$ is the line $y = 2x + 3$, which is not a linear function since it does not pass through the origin. To rectify this rather minor shortcoming we first translate the graph of the function f so that the point $(a, f(a))$ goes to the origin, and *then* find the best linear approximation at the origin. The operation of translation is carried out by a somewhat notorious acquaintance from beginning calculus Δy. The source of its notoriety is two-fold: first, in many texts it is inadequately defined; and second, the notation Δy fails to alert the reader to the fact that under consideration is a function of *two* variables. We will be careful on both counts.

8.3.1. Definition. Let $f \in \mathcal{F}_a$. Define the function Δf_a by

$$\Delta f_a(h) := f(a + h) - f(a)$$

for all h such that $a + h$ is in the domain of f. Notice that since f is defined in a neighborhood of a, the function Δf_a is defined in a neighborhood of 0; that is, Δf_a belongs to $\mathcal{F}_0$. Notice also that $\Delta f_a(0) = 0$.

8.3.2. Problem. Let $f(x) = \cos x$ for $0 \le x \le 2\pi$.

(a) Sketch the graph of the function f.
(b) Sketch the graph of the function Δf_π.

8.3.3. Proposition. *If $f \in \mathcal{F}_a$ and $\alpha \in \mathbb{R}$, then*

$$\Delta(\alpha f)_a = \alpha\,\Delta f_a\,.$$

PROOF. To show that two functions are equal show that they agree at each point in their domain. Here

$$\begin{aligned}\Delta(\alpha f)_a(h) &= (\alpha f)(a + h) - (\alpha f)(a)\\ &= \alpha f(a + h) - \alpha f(a)\\ &= \alpha(f(a + h) - f(a))\\ &= \alpha\,\Delta f_a(h)\end{aligned}$$

for every h in the domain of Δf_a. □