8.3.4. Proposition. *If $f, g \in \mathcal{F}_a$, then*

$$\Delta(f+g)_a = \Delta f_a + \Delta g_a\,.$$

PROOF. Exercise. (Solution Q.8.8.)

The last two propositions prefigure the fact that differentiation is a linear operator; the next result will lead to *Leibniz's rule* for differentiating products.

8.3.5. Proposition. *If $\phi, f \in \mathcal{F}_a$, then*

$$\Delta(\phi f)_a = \phi(a)\cdot\Delta f_a \,+\, \Delta\phi_a \cdot f(a) \,+\, \Delta\phi_a\cdot\Delta f_a\,.$$

PROOF. Problem.

Finally, we present a result which prepares the way for the *chain rule*.

8.3.6. Proposition. *If $f \in \mathcal{F}_a$, $g \in \mathcal{F}_{f(a)}$, and $g\circ f \in \mathcal{F}_a$, then*

$$\Delta(g\circ f)_a = \Delta g_{f(a)} \circ \Delta f_a\,.$$

PROOF. Exercise. (Solution Q.8.9.)

8.3.7. Proposition. *Let $A \subseteq \mathbb{R}$. A function $f\colon A \to \mathbb{R}$ is continuous at the point a in A if and only if Δf_a is continuous at 0.*

PROOF. Problem.

8.3.8. Proposition. *If $f\colon U \to U_1$ is a bijection between subsets of $\mathbb{R}$, then for each a in U the function $\Delta f_a\colon U - a \to U_1 - f(a)$ is invertible and*

$$\big(\Delta f_a\big)^{-1} = \Delta\big(f^{-1}\big)_{f(a)}\,.$$

PROOF. Problem.

8.4. DIFFERENTIABILITY

We now have developed enough machinery to talk sensibly about *differentiating* real valued functions.

8.4.1. Definition. Let $f \in \mathcal{F}_a$. We say that f is DIFFERENTIABLE AT a if there exists a linear function which is tangent at 0 to Δf_a. If such a function exists, it is called the DIFFERENTIAL of f at a and is denoted by df_a. (Don't be put off by the slightly complicated notation; df_a is just a member of $\mathfrak{L}$ satisfying $df_a \simeq \Delta f_a$.) We denote by $\mathcal{D}_a$ the family of all functions in $\mathcal{F}_a$ which are differentiable at a.

The next proposition justifies the use of the definite article which modifies "differential" in the preceding paragraph.

8.4.2. Proposition. *Let $f \in \mathcal{F}_a$. If f is differentiable at a, then its differential is unique. (That is, there is at most one linear map tangent at 0 to Δf_a.)*

PROOF. Proposition 8.2.5. □

8.4.3. Example. It is instructive to examine the relationship between the differential of f at a, which we defined in 8.4.1, and the derivative of f at a as defined in beginning calculus. For $f \in \mathcal{F}_a$ to be differentiable at a it is necessary that there be a linear function $T\colon \mathbb{R} \to \mathbb{R}$ which is tangent at 0 to Δf_a. According to 8.1.7 there must exist a constant c such that $Tx = cx$ for all x in $\mathbb{R}$. For T to be tangent to Δf_a, it must be the case that

$$\Delta f_a - T \in \mathfrak{o}\,;$$

that is,

$$\lim_{h\to 0}\frac{\Delta f_a(h) - ch}{h} = 0\,.$$