Metric Space

[] Let $\mathbb{X}$ be any unstructured nonempty set.

[] Let $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ be a *function* that assigns a real number to any pair of elements in the set $\mathbb{X}$. Intuitively, it takes a pair of elements in the set $\mathbb{X}$ as the input and provides the distance between them (some real number) as the output. We could also say that d takes an element from the set $\mathbb{X} \times \mathbb{X}$ as the input and produces a real number as the output.

[] The tuple $(\mathbb{X}, d)$ is called a *metric space* if the following three conditions hold.

- *Non-negativity*: $d(x, y) \geq 0 \; \forall x, y \in \mathbb{X}$, and $d(x, y) = 0$ iff $x = y$.
- *Symmetry*: $d(x, y) = d(y, x) \; \forall x, y \in \mathbb{X}$.
- *Triangular inequality*: $d(x, y) \leq d(x, z) + d(z, y) \; \forall x, y, z \in \mathbb{X}$.

[] The function d is called *a metric on the set* $\mathbb{X}$.

[] Unless necessary, we will denote the metric space as $\mathbb{X}$.

[] *ε-neighborhood of x in* $\mathbb{X}$: For any $x \in \mathbb{X}$ and $\varepsilon \in \mathbb{R}_{++}$, the *$\varepsilon$-neighborhood of x in X* is

- the set $N_{\varepsilon,\mathbb{X}}(x) = \{y \in \mathbb{X} : d(x, y) < \varepsilon\}$.

[] *Open subset of* $\mathbb{X}$: A subset S of $\mathbb{X}$ is said to be *open in* $\mathbb{X}$ if

- for each $x \in S$,
- $\exists \; \varepsilon \in \mathbb{R}_{++}$ such that,
- $N_{\varepsilon,\mathbb{X}}(x) \subseteq S$.

[] *Closed subset of* $\mathbb{X}$: A subset S of $\mathbb{X}$ is said to be *closed in* $\mathbb{X}$ if

- $\mathbb{X} \backslash S$ is open in $\mathbb{X}$.

[] $S \subset \mathbb{X}$ may be open, not open, closed, not closed, clopen, or neither open nor closed.

[] *Closure of a subset of* $\mathbb{X}$: Given any $S \subseteq X$, the closure of S in $\mathbb{X}$ is

- the smallest closed set $Cl_X(S)$ such that
- $S \subseteq Cl_X(S)$.