

### Example

**Theorem 1.5.1.** *If  $0 \leq x \leq 2$ , then  $-x^3 + 4x + 1 > 0$ .*

Before we write a proof of this theorem, we have to do some scratchwork to figure out why it is true.

The inequality certainly holds for  $x = 0$ ; then the left side is equal to 1 and  $1 > 0$ . As  $x$  grows, the  $4x$  term (which is positive) initially seems to have greater magnitude than  $-x^3$  (which is negative). For example, when  $x = 1$ , we have  $4x = 4$ , but  $-x^3 = -1$  only. In fact, it looks like  $-x^3$  doesn't begin to dominate until  $x > 2$ . So it seems the  $-x^3 + 4x$  part should be nonnegative for all  $x$  between 0 and 2, which would imply that  $-x^3 + 4x + 1$  is positive.

So far, so good. But we still have to replace all those “seems like” phrases with solid, logical arguments. We can get a better handle on the critical  $-x^3 + 4x$  part by factoring it, which is not too hard:

$$-x^3 + 4x = x(2 - x)(2 + x)$$

Aha! For  $x$  between 0 and 2, all of the terms on the right side are nonnegative. And a product of nonnegative terms is also nonnegative. Let's organize this blizzard of observations into a clean proof.

*Proof.* Assume  $0 \leq x \leq 2$ . Then  $x$ ,  $2 - x$ , and  $2 + x$  are all nonnegative. Therefore, the product of these terms is also nonnegative. Adding 1 to this product gives a positive number, so:

$$x(2 - x)(2 + x) + 1 > 0$$

Multiplying out on the left side proves that

$$-x^3 + 4x + 1 > 0$$

as claimed. ■

There are a couple points here that apply to all proofs:

- You'll often need to do some scratchwork while you're trying to figure out the logical steps of a proof. Your scratchwork can be as disorganized as you like—full of dead-ends, strange diagrams, obscene words, whatever. But keep your scratchwork separate from your final proof, which should be clear and concise.
- Proofs typically begin with the word “Proof” and end with some sort of delimiter like  $\square$  or “QED.” The only purpose for these conventions is to clarify where proofs begin and end.