From the figure above, $\quad \mathrm{pr}\{0 < X + Y < z\} = \begin{cases} \frac{1}{2}z^2 & \text{if } 0 < z < 1, \\ 1 - \frac{1}{2}(2-z)^2 & \text{if } 1 \le z < 2. \end{cases}$

An alternative derivation uses integration. For example, in the case $z < 1$,

$$\mathrm{pr}\{0 < X + Y < z\} = \iint_{0<x+y<z} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{y=0}^{z}\int_{x=0}^{z-y} \mathrm{d}x\mathrm{d}y = \int_{y=0}^{z}(z-y)\mathrm{d}y = \frac{1}{2}z^2.$$

(b) If $Z = X + Y$, then Z has cumulative distribution function $F(z)$ where, from (a) above,

$$F(z) = \mathrm{pr}\{X + Y \le z\} = \begin{cases} \frac{1}{2}z^2 & \text{if } 0 < z < 1, \\ 1 - \frac{1}{2}(2-z)^2 & \text{if } 1 \le z < 2. \end{cases}$$

Probability density function for Z is then $f(z) = \dfrac{\mathrm{d}F(z)}{\mathrm{d}z} = \begin{cases} z & \text{if } 0 < z < 1, \\ 2 - z & \text{if } 1 \le z < 2. \end{cases}$

Z has a triangular distribution on the interval $(0, 2)$ with mode at $z = 1$.

Worked Example: Lecture 14

Measurements of stature were made on each member of a large population of pairs of adult brothers. The height of the elder brother was denoted by X and of the younger brother by Y. Both X and Y had the same mean μ and the same standard deviation σ. The correlation coefficient was ρ.
Deduce the mean and variance of (i) $U = X - Y$, and (ii) $V = X + Y$.
Derive the covariance of U and V.

Answer:

$$\mathrm{E}[U] = \mathrm{E}[X - Y] = \mathrm{E}[X] - \mathrm{E}[Y] = \mu - \mu = 0.$$

$$\mathrm{Var}[U] = \mathrm{Var}[X - Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] - 2\mathrm{cov}(X, Y) = \sigma^2 + \sigma^2 - 2\rho\sigma^2 = 2\sigma^2(1 - \rho),$$

as

$$\mathrm{corr}(X, Y) = \frac{\mathrm{cov}(X, Y)}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}} \quad \Rightarrow \quad \mathrm{cov}(X, Y) = \mathrm{corr}(X, Y)\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]} = \rho\sqrt{\sigma^2\sigma^2} = \rho\sigma^2.$$

$$\mathrm{E}[V] = \mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y] = \mu + \mu = 2\mu.$$

$$\mathrm{Var}[V] = \mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{cov}(X, Y) = \sigma^2 + \sigma^2 + 2\rho\sigma^2 = 2\sigma^2(1 + \rho).$$

$$\begin{aligned} \mathrm{cov}(U, V) = \mathrm{cov}(X - Y, X + Y) &= \mathrm{cov}(X, X) - \mathrm{cov}(Y, Y) + \mathrm{cov}(X, Y) - \mathrm{cov}(Y, X) \\ &= \mathrm{Var}[X] - \mathrm{Var}[Y] + \mathrm{cov}(X, Y) - \mathrm{cov}(X, Y) \\ &= \sigma^2 - \sigma^2 + \mathrm{cov}(X, Y) - \mathrm{cov}(X, Y) \\ &= 0. \end{aligned}$$

Worked Example: Lecture 14.

Let $T = a_1X_1 + a_2X_2$, where X_1 and X_2 are uncorrelated random variables with mean μ and variance σ^2, and a_1 and a_2 are constants chosen so that $\mathrm{E}[T] = \mu$.
Deduce that the choice $a_2 = 1 - a_1$ gives $E[T] = \mu$.
In this case prove that the variance of T is a minimum if $a_1 = a_2 = \frac{1}{2}$.