Answer:

$$\mathrm{E}[T] = E[a_1X_1 + a_2X_2] = a_1\mathrm{E}[X_1] + a_2\mathrm{E}[X_2] = a_1\mu + a_2\mu = (a_1 + a_2)\mu.$$

If we require $E[T] = \mu$, then $a_1 + a_2 = 1$, so that $a_2 = 1 - a_1$.
Since $E[T] = \mu$, then T is said to be an unbiased estimator *of the mean μ.*

$$\mathrm{Var}[T] = \mathrm{Var}[a_1X_1 + a_2X_2] = a_1^2\mathrm{Var}[X_1] + a_2^2\mathrm{Var}[X_2] = a_1^2\sigma^2 + a_2^2\sigma^2 = (a_1^2 + a_2^2)\sigma^2.$$

Since $a_2 = 1 - a_1$, $\mathrm{Var}[T] = \{a_1^2 + (1 - a_1)^2\}\sigma^2 = (2a_1^2 - 2a_1 + 1)\sigma^2$. Differentiate this with respect to a_1 to find the minimum.

$$\frac{\mathrm{d}}{\mathrm{d}a_1}\mathrm{Var}[T] = (4a_1 - 2)\sigma^2,$$

which is zero when $a_1 = \frac{1}{2}$. Hence $\mathrm{Var}[T]$ is a minimum when $a_1 = a_2 = \frac{1}{2}$ so $T = \frac{1}{2}(X_1 + X_2)$.
Alternative derivation: write $a_1 = \frac{1}{2} + \varepsilon$, $a_2 = \frac{1}{2} - \varepsilon$. Then

$$\mathrm{Var}[T] = (a_1^2 + a_2^2)\sigma^2 = \{(\tfrac{1}{2} + \varepsilon)^2 + (\tfrac{1}{2} - \varepsilon)^2\}\sigma^2 = (\tfrac{1}{2} + 2\varepsilon^2)\sigma^2,$$

and is a minimum if $\varepsilon = 0$.

What does this question show? In part (a) you chose a_2 to restrict attention to linear combinations of the X_i which were unbiased estimators of the mean μ, so $\mathrm{E}[T] = \mu$. In part (b) you then showed that of all such unbiased estimators, the sample mean $\bar{X}$ is the one with smallest variance, so giving values closest to the true mean μ.

Worked Example: Lecture 15.

The following data give the noise level (in decibels) generated by fourteen different chain saws powered in one of two different ways.

Petrol-powered chain saws	103	103	105	106	108	105	106
Electric-powered chain saws	97	95	94	93	91	95	94

At the 5% level of significance, test whether the average noise level of petrol-powered chain saws is higher than for electric-powered chain saws.

Answer: Testing $\mathrm{H}_0 : \mu_1 = \mu_2$ *vs.* $\mathrm{H}_1 : \mu_1 > \mu_2$, *i.e.* $\mathrm{H}_0 : \mu_1 - \mu_2 = 0$ *vs.* $\mathrm{H}_1 : \mu_1 - \mu_2 > 0$. Have two independent samples with unknown variance. Need to assume variances are equal.

Worked Example: Lecture 15.

The following data give the length (in mm.) of cuckoo (*cuculus canorus*) eggs found in nests belonging to wrens (A) and reed warblers (B).

A:	19.8	22.1	21.5	20.9	22.0	21.0	22.3	21.0	20.3	20.9
B:	23.2	22.0	22.2	21.2	21.6	21.6	21.9	22.0	22.9	22.8

Assuming the variances for each group are the same, is there any evidence at the 5% level to suggest that the egg size differs between the two host species?