$$Z = \frac{S^2}{\rho k} T_a = \frac{\left(\frac{Volt}{K}\right)^2}{(ohm\ m)\left(\frac{W}{mK}\right)} K = \frac{\left(\frac{J/coul}{K}\right)^2}{\left(\frac{Js}{coul^2} m\right)\left(\frac{(J/s)}{mK}\right)} K = \frac{J^2}{J^2} \frac{\frac{1}{coul^2}}{\frac{1}{coul^2}} \frac{1}{s(1/s)} \frac{\frac{1}{K^2} K}{\frac{1}{K}} = 1 \text{ OK}$$

(b) show that the equation makes physical sense (passes function test)

- If the material Z = 0, it produces no electrical power thus the efficiency should be zero. If Z = 0 then

$$\eta = \left(1 - \frac{T_L}{T_H}\right) \frac{\sqrt{1 + 0T_a} - 1}{\sqrt{1 + 0T_a} + T_L/T_H} = \left(1 - \frac{T_L}{T_H}\right) \frac{\sqrt{1} - 1}{\sqrt{1} + T_L/T_H} = \left(1 - \frac{T_L}{T_H}\right) \frac{0}{1 + T_L/T_H} = 0 \qquad \text{OK}$$

- If $T_L = T_H$, then there is no temperature difference across the thermoelectric material, and thus no power can be generated. In this case

$$\eta = (1 - 1) \frac{\sqrt{1 + ZT_a} - 1}{\sqrt{1 + ZT_a} + 1} = (0) \frac{\sqrt{1 + ZT_a} - 1}{\sqrt{1 + ZT_a} + 1} = 0 \qquad \text{OK}$$

- Even the best possible material ($ZT_a \to \infty$) cannot produce an efficiency greater than the theoretically best possible efficiency (called the *Carnot cycle* efficiency, see page 88) = 1 – T_L/T_H, for the same temperature range. As $ZT_a \to \infty$,

$$\eta \approx \left(1 - \frac{T_L}{T_H}\right) \frac{\sqrt{ZT_a} - 1}{\sqrt{ZT_a} + T_L/T_H} \approx \left(1 - \frac{T_L}{T_H}\right) \frac{\sqrt{ZT_a}}{\sqrt{ZT_a}} = 1 - \frac{T_L}{T_H} \qquad \text{OK}$$

Side note #1: a good thermoelectric material such as Bi_2Te_3 has $ZT_a \approx 1$ and works up to about 200°C before it starts to melt, thus

$$\eta = \left(1 - \frac{T_L}{T_H}\right) \frac{\sqrt{1+1} - 1}{\sqrt{1+1} + (25+273)/(200+273)} = 0.203\left(1 - \frac{T_L}{T_H}\right) = 0.203\eta_{Carnot}$$

$$= 0.203\left(1 - \frac{25+273}{200+273}\right) = 0.0750 = 7.50\%$$

By comparison, your car engine has an efficiency of about 25%. So practical thermoelectric materials are, in general, not very good sources of electrical power, but are extremely useful in some niche applications, particularly when either (1) it is essential to have a device with no moving parts or (2) a "free" source of thermal energy at relatively low temperature is available, e.g. the exhaust of an internal combustion engine.

Side note #2: a good thermoelectric material has a high S, so produces a large voltage for a small temperature change, a low ρ so that the resistance of the material to the flow of electric current is